

THE COORDINATIZATION OF ARGUESIAN LATTICES

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ABSTRACT. We show that the auxiliary planar ternary ring of an n -frame in an Arguesian lattice, $n \geq 3$, is indeed an associative ring with unit. The addition of two weak necessary conditions allows us to coordinatize a hyperplane of this n -frame. This generalizes the classical work of von Neumann, Baer-Inaba, Jónsson and Jónsson-Monk.

1. Introduction. The classical coordinatization theorem for (Desarguean) projective geometries was vastly extended by von Neumann in [17] where he showed that every complemented modular lattice of order $n \geq 4$ was isomorphic to the lattice of principal (left) ideals of a regular ring. An analogous coordinatization result was proven by Baer [2] and Inaba [8] for primary modular lattices of order $n \geq 4$. They showed that any such lattice was isomorphic to the lattice (!) of finitely generated submodules of R^n for some completely primary, uniserial ring R .

In [10], Jónsson introduced a lattice identity strictly stronger than the modular identity which reflected the Desarguean axiom of projective geometry. He properly called this identity the Arguesian law and in a series of (sometimes joint) papers, Jónsson (et al.) proved many important consequences of it (see the bibliography). In particular, Jónsson [11] extended von Neumann's result to complemented Arguesian lattices of order $n \geq 3$, and Jónsson and Monk [15] extended the Baer-Inaba result to primary Arguesian lattices of order $n \geq 3$.

Since all submodule lattices are indeed Arguesian (Jónsson [10]), von Neumann and Baer-Inaba's deep results tell us that certain modular lattices of order $n \geq 4$ are Arguesian. However, one needs only Frink [4] (as presented in Crawley and Dilworth [3, Theorem 13.1, p. 105]) and G. S. Monk [16] to obtain these results without any heavy coordinatization machinery. It is reasonable then to start with an Arguesian lattice of order $n \geq 3$ and develop the theory of coordinatization from there.

2. Preliminaries. A lattice, $(L; +, \cdot)$, is called *Arguesian* [10] if it satisfies the six variable equation

$$(\arg): (a_0 + b_0)(a_1 + b_1)(a_2 + b_2) \leq a_0(a_1 + c) + b_0(b_1 + c)$$

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where $c = c_2(c_0 + c_1)$ and $c_i = (a_j + a_k)(b_j + b_k)$, $\{i, j, k\} = \{0, 1, 2\}$. (Strictly speaking this is not an equation but modulo the theory of lattices, $p \leq q$ if and only if $p = p \cdot q$ if and only if $p + q = q$.) That this equation models Desargues's axiom requires some more definitions. A *triangle in L* is a triple $\mathbf{a} = (a_0, a_1, a_2) \in L^3$; two triangles \mathbf{a}, \mathbf{b} in L are called *centrally perspective* if $(a_0 + b_0)(a_1 + b_1) \leq a_2 + b_2$ and *axially perspective* if $c_2(\mathbf{a}, \mathbf{b}) \leq c_0(\mathbf{a}, \mathbf{b}) + c_1(\mathbf{a}, \mathbf{b})$ where, as above, $c_i(\mathbf{a}, \mathbf{b}) = c_i = (a_j + a_k)(b_j + b_k)$, $\{i, j, k\} = \{0, 1, 2\}$. A lattice, $(L; +, \cdot)$, is called *Desarguean* if any centrally perspective pair of triangles in L is also axially perspective. The following results are in Jónsson et al. [5, 14 and 15].

(2.1) THEOREM. (1) *Arguesian (resp. Desarguean) lattices are modular.*

(2) *A lattice is Arguesian if and only if it is Desarguean.*

(3) *The dual of an Arguesian lattice is again Arguesian.*

(4) *For triangles \mathbf{a}, \mathbf{b} in L , an Arguesian lattice, \mathbf{a} and \mathbf{b} are axially perspective if and only if $(a_0 + b_0)(a_1 + b_1) \leq (a_0 + a_2)(a_1 + a_2) + (b_0 + b_2)(b_1 + b_2)$.*

We will call two triangles \mathbf{a}, \mathbf{b} in a lattice L *doubly centrally perspective* (or just doubly CP) if both of the pairs (\mathbf{a}, \mathbf{b}) and $(a_0, a_2, a_1), (b_0, b_2, b_1)$ are CP. In an Arguesian lattice, this will imply that $c_2 + c_0 = c_1 + c_0$. They will be called *triply centrally perspective* if they are CP for any permutation of the indices. In an Arguesian lattice this will (of course) imply $c_2 + c_1 = c_2 + c_0 = c_1 + c_0$. Finally \mathbf{a} and \mathbf{b} will be called *normal* if $a_2 = (a_0 + a_2)(a_1 + a_2)$ and $b_2 = (b_0 + b_2)(b_1 + b_2)$. From Theorem 2.1(4) we obtain that normal triangles are CP if and only if they are AP (= axially perspective).

In our coordinatization theory we use Huhn's notion of an n -diamond rather than the (definitionally equivalent) notion of a (homogeneous) n -frame due to von Neumann. In §4, we will discuss their equivalence and examine our results in the von Neumann setting. We feel that the n -diamond approach more naturally generalizes the classical geometric approach. Hopefully the reader will agree.

An n -diamond in a (modular) lattice, L , is a sequence $\mathbf{d} = (d_1, \dots, d_{n+1})$ in L^{n+1} satisfying

$$(nD1) \quad v = \sum(d_j : j \neq i), \quad \text{all } i$$

and

$$(nD2) \quad u = d_i \cdot \sum(d_k : k \neq i, j), \quad \text{all } i \neq j.$$

If $u = 0_L$ and $v = 1_L$, \mathbf{d} is called a *spanning n -diamond*.

A spanning n -diamond formalizes the idea of $(n - 1) + 2$ points in general position in a projective geometry of (projective) dimension $(n - 1)$. We have called the concept an n -diamond (rather than an $(n - 1)$ -diamond as in Huhn [7]) in order to make the natural number agree with the homogeneous dimension of coordinatizable projective geometries ($\text{PG}_{n-1}(K) \simeq \mathcal{L}(K^n)$) and with von Neumann's (same natural number)-frame.

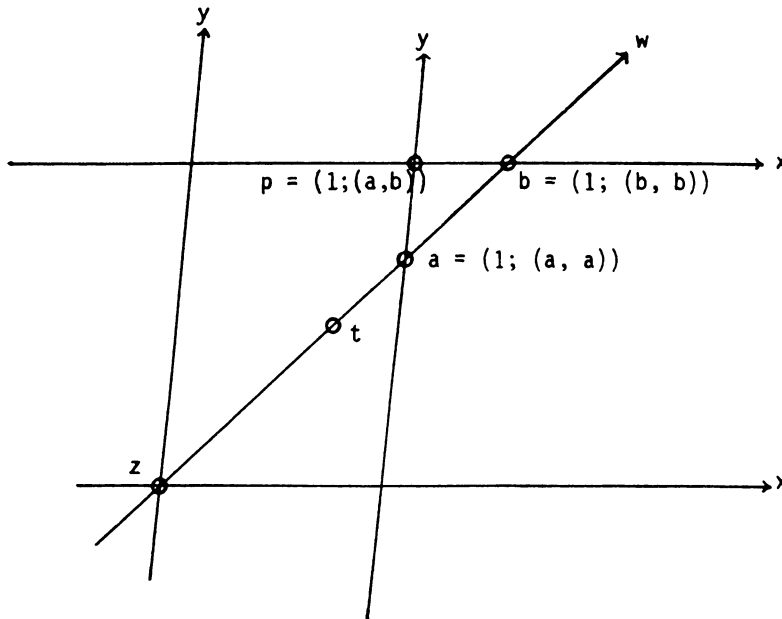
If $(L; +, \cdot)$ is a modular lattice with a spanning n -diamond then we say L is of *order n* and denote the particular spanning n -diamond very asymmetrically as $\mathbf{x} = (x_1, \dots, x_{n-1}, z, t)$. We define (and think of) $h_\infty = \sum(x_i : 1 \leq i \leq n - 1)$ as the

hyperplane at infinity; $A = \{p \in L: p + h_\infty = 1 \text{ and } p \cdot h_\infty = 0\}$ as the associated affine space; $w = h_\infty(z + t)$ as the infinity point on the line $z + t$; $D = \{a \in A: a \leq z + t\} = \{p \in L: p + w = z + t \text{ and } p \cdot w = 0\}$ as the affine points on the diagonal $z + t$. Naturally all of the above definitions depend on the given n -diamond, x .

The above definitions allow one to "coordinatize" the "affine plane" A in that there is a natural bijection $A \rightarrow D^{n-1}$ given by: $p \rightsquigarrow ((z + t)(\bar{x}_i + p))$ where $\bar{x}_i = \Sigma(x_j: j \neq i)$. The inverse of the above map is the function $a \rightsquigarrow \Pi(\bar{x}_i + a_i)$. We will abbreviate this last expression as $(1; a)$. Thus $A = \{(1; a): a \in D^{n-1}\}$.

We illustrate with two examples.

(2.2) EXAMPLE A. A projective plane can be considered as an affine plane with a line at infinity.



Here (x, y, z, t) is the spanning 3-diamond with $h_\infty = x + y = x + w = y + w$, the line at infinity.

(2.3) EXAMPLE B. The (Arguesian) lattice $\mathcal{L}(R^n)$.

Let $\{e_0, e_1, \dots, e_{n-1}\}$ be the standard basis for the (free) left R -module, ${}_R R^n$. We define $x_1 = Re_1, \dots, x_{n-1} = Re_{n-1}$, $z = Re_0$ and $t = R(e_0 + \Sigma(e_i: 1 \leq i \leq n-1))$. Then

- (i) $h_\infty = \langle e_1, \dots, e_{n-1} \rangle$, the submodule generated by $\{e_1, \dots, e_{n-1}\}$,
- (ii) $A = \{R(e_0 + \Sigma a_i e_i): a \in R^{n-1}\}$,
- (iii) $w = R(\Sigma e_i)$,
- (iv) $D = \{R(e_0 + a(\Sigma e_i)): a \in R\}$.

If (x, y, z, t) is a spanning 3-diamond in a modular lattice, L , then the projective isomorphism $[0, z + t] \xrightarrow{x} [0, y + z]$ maps D onto $D_0 = \{p \in A: p + y = y + z \text{ and } py = 0\}$ by $b \xrightarrow{x} b_0 = (y + z)(x + b)$. These are the Y -intercept points with coordinates $(1; (0, b))$. We define $v = t_0$.

Similarly the two-step projective isomorphism $[0, z + t] \stackrel{x}{\wedge} [0, y + t] \stackrel{z}{\wedge} [0, x + y]$ mapping $b \stackrel{x}{\wedge} b_1 \stackrel{z}{\wedge} b_\infty$ defines the set of slope points at infinity $D_\infty = \{q \in L: q + y = x + y \text{ and } qy = 0\}$ via the set $D_1 = \{r \in A: r + y = y + t \text{ and } ry = 0\}$. These “points” allow us to define the *ternary operator, addition and multiplication* on D by

$$\begin{aligned} T(a, m, b) &= (z + t)(x + (y + a)(m_\infty + b_0)), \\ a \oplus b &= T(a, t, b) = (z + t)(x + (y + a)(w + b_0)), \\ a \otimes b &= T(a, b, z) = (z + t)(x + (y + a)(z + b_1)). \end{aligned}$$

Since L is modular, these are indeed operations on D . In Example A they are the classical operations and in Example B, with $n = 3$ and $\bar{a} = R(e_0 + a(e_1 + e_2))$, $T(\bar{a}, \bar{m}, \bar{b}) = \bar{a} \times \bar{m} + \bar{b}$.

Finally if $\mathbf{x} = (x_1, \dots, x_{n-1}, z, t)$ is a spanning n -diamond in a modular lattice L with $n \geq 4$, then we may form $(n - 1)$ 3-diamonds (x_i, y_i, z, t) spanning $[0, z + t + x_i]$ where $y_i = \bar{x}_i(z + t + x_i)$. All of these 3-diamonds have the same diagonal, D , and diagonal point at infinity, w . These produce $(n - 1)$ ternary operators $T_i: D^3 \rightarrow D$. Now let $P = \{(p, q): w = (p + q)(z + t) \text{ and } p(z + t) = q(z + t) = 0\}$. Each $(p, q) \in P$ defines a ternary operator³ on D by

$$\begin{aligned} T_{(p,q)}(a, m, b) \\ = (z + t)(p + (q + a)[(p + q)(z + (q + t)(p + m)) + (q + z)(p + b)]). \end{aligned}$$

Since L is modular, if $(p, q) \lessgtr (r, s)$ (i.e. are comparable in (P, \leq)), $T_{(p,q)} = T_{(r,s)}$. Now (x_i, y_i) , (x_i, \bar{x}_i) , (y_i, x_i) , (\bar{x}_i, x_i) and (\bar{x}_i, \bar{x}_i) for $i \neq j$ are all in P . Since $n - 1 \geq 3$, the ternary operators produced by all of the above pairs are identical. This also supplies a unique definition of multiplication and addition.

In the following we will define other functions using a 3-diamond $\langle x, y, z, t \rangle$ and D . The above discussion will apply to these functions as well if a larger n -diamond is present.

(2.4) LEMMA. *Let L be a modular lattice of order n , then $(D; \oplus, z)$ is a loop with left and right difference operations defined by*

$$\begin{aligned} c\Delta_r b &= (z + t)(y + (x + c)(w + b_0)), \\ a_l\Delta c &= (z + t)(x + (y + z)(w + (y + a)(x + c))) \end{aligned}$$

(i.e. $c = a \oplus b$ iff $a = c\Delta_r b$ iff $b = a_l\Delta c$).

In general, multiplication is not so well behaved. We define $\text{Inv}(D) = \{a \in D: z + a = z + t \text{ and } z \cdot a = 0\}$ and two divisibility functions from D^2 into $[0, z + t]$ by

$$\begin{aligned} c/b &= (z + t)(y + (x + c)(z + b_1)), \\ a \setminus c &= (z + t)(x + (y + t)(z + (y + a)(x + c))). \end{aligned}$$

³See Appendix 1.

(2.5) LEMMA. Let L be a modular lattice of order n ; then:

(1) t is the unit and z is the zero of \otimes .

(2) If $a \in \text{Inv}(D)$, $a \setminus b$ and $b/a \in D$.

(3)⁴ $a \in \text{Inv}(D)$ iff there exists $b, c \in D$ with $b \otimes a = t = a \otimes c$.

(4) For $a \in D$, the functor $a \otimes -: [0, z + t] \rightarrow [0, z + t]$ has as left adjoint $a \setminus -$. That is, for all $p, q \in [0, z + t]$, $a \setminus p \leq q$ iff $p \leq a \otimes q$.

(5) For $b \in D$, the functor $- \otimes b: [0, z + t] \rightarrow [0, z + t]$ has a right adjoint $- / b$. That is, for all $p, q \in [0, z + t]$, $p \otimes b \leq q$ iff $p \leq q / b$.

We close by noting that in Example B, $\bar{a} \setminus \bar{c} = R(ae_0 + c(\sum e_i))$ and $\bar{c} / \bar{b} = \{xe_0 + y\sum e_i: xc = yb\}$.

3. The auxiliary ring $(D; \oplus, z, \otimes, t)$. In this section we will assume L is an Arguesian lattice of order 3 with a fixed spanning 3-diamond (x, y, z, t) , and show that the associated “diagonal” D is a ring under the operations defined in §2. By using Example A and a standard text on projective geometry, the reader will notice that some of the initial lemmata are well known. This is not always true as geometric proofs can assume that the meet of distinct points is 0. In some cases, we have found only slightly more circuitous proofs. Our proof that \oplus is commutative, however, requires essential use of the multiplicative unit, t . This is definitely not required in Example A and, it seems to us, should be a flaw. We (obviously) know no other approach.

(3.1) LEMMA. The ternary operator $T: D^3 \rightarrow D$ is linear (i.e. $T(a, b, c) = (a \otimes b) \oplus c$).

PROOF. The triangles $\langle y, c_0, (y + a)(b_\infty + c_0) \rangle$ and $\langle a \otimes b, w, x \rangle$ are normal and their CP statement is equivalent (via modularity) to our claim. They are, however, axially perspective since

$$\begin{aligned} [y + (y + a)(b_\infty + c_0)][a \otimes b + x] + [c_0(y + a)(b_\infty + c_0)][w + x] \\ = (z + b_1)(y + z + a) \end{aligned}$$

and

$$[y + c_0][a \otimes b + w] = z. \quad \square$$

(3.2) LEMMA. \otimes is left distributive over \oplus .

PROOF. In light of (3.1), we need only show $a \otimes T(t, b, c) = T(a, b, a \otimes c)$. This is equivalent to showing $(y + a)(z + (y + t)(b_\infty + c_0)) \leq b_\infty + (a \otimes c)_0$. Since the above is the CP statement for the normal triangles $\langle (y + a)(x + a \otimes c), z, (a \otimes c)_0 \rangle$ and $\langle y, (y + t)(b_\infty + c_0), b_\infty \rangle$, we need only check their axial perspectivity. For

$$q_2 = [(y + a)(x + a \otimes c) + z][y + (y + t)(b_\infty + c_0)] = c_1(y + z + a),$$

$$q_1 = [(y + a)(x + a \otimes c) + (a \otimes c)_0][y + b_\infty] = x(y + z + a),$$

$$q_0 = [z + (a \otimes c)_0][(y + t)(b_\infty + c_0) + b_\infty] = c_0(x + z + c_1(y + z + a)),$$

we obtain $q_0 + q_1 = (x + c_0)(y + z + a)(x + z + q_2) \geq q_2$. \square

⁴See Appendix 2.

(3.3) LEMMA. \otimes is right distributive over \oplus .

PROOF. Again in light of (3.1), it is enough to show $(a \oplus b) \otimes c = T(a, c, b \otimes c)$. Moreover, $T(a, c, b \otimes c) \leq (a \oplus b) \otimes c$ iff $x + (y + a)(c_\infty + (b \otimes c)_0) \leq x + (y + a \oplus b)(z + c_\infty)$ iff $(y + a \oplus b)(x + (y + a)(c_\infty + (b \otimes c)_0)) \leq z + c_\infty$. This last inequality is precisely the CP statement for the normal triangles $\langle x, a \oplus b, z \rangle$ and $\langle (y + a)(c_\infty + (b \otimes c)_0), y, c_\infty \rangle$. The equivalent AP statement for these triangles is

$$\begin{aligned} (y + a)(w + b_0) &\leq w(x + z + (y + a)(w + b_0)) + (x + z)(c_\infty + (b \otimes c)_0) \\ &\text{iff } (y + a)(w + b_0) \leq w + (x + z)(c_\infty + (b \otimes c)_0) \\ &\text{iff } b_0 \leq w + (x + z)(c_\infty + (b \otimes c)_0). \end{aligned}$$

This inequality is easily implied by the AP statement for the triangles $\langle x, b, z \rangle$ and $\langle (b \otimes c)_0, y, c_\infty \rangle$ and these triangles are indeed centrally perspective. Therefore multiplication is right distributive over addition.

(3.4) LEMMA. For any $\bar{x}, \bar{y} \leq h$ with $\bar{x} + w = \bar{y} + w = \bar{x} + \bar{y} = h$ and $\bar{x} \cdot w = \bar{y} \cdot w = \bar{x} \cdot \bar{y} = 0$,

$$a \otimes b = (z + t)(\bar{x} + (\bar{y} + a)(z + (\bar{y} + t)(\bar{x} + b))).$$

PROOF. The given properties for \bar{x} and \bar{y} are sufficient to define a “multiplication” $\bar{\otimes}: D^2 \rightarrow D$ for which $a \bar{\otimes} b \leq z + b$. This multiplication is independent of the \bar{y} since the triangles $\langle a, \bar{y}_1, \bar{y}_2 \rangle$ and $\langle z, (\bar{y}_1 + t)(\bar{x} + b), (\bar{y}_2 + t)(\bar{x} + b) \rangle$ are (doubly) centrally perspective at $t(z + a)$. This multiplication is also independent of the \bar{x} by considering the (doubly) normal triangles

$$\langle b, \bar{x}_1, \bar{x}_2 \rangle$$

and

$$\langle z, (\bar{y} + a)(z + (\bar{y} + t)(\bar{x}_1 + b)), (\bar{y} + a)(z + (\bar{y} + t)(\bar{x}_2 + b)) \rangle.$$

Now for a given $\bar{x}, \bar{y}, e = (y + t)(w + v)$ and $\bar{e} = (\bar{y} + t)(w + v)$, the normal triangles $\langle z, (\bar{y} + a)(z + \bar{e}), (\bar{y} + a)(z + (\bar{y} + t)(\bar{x} + b)) \rangle$ and $\langle b, h(\bar{e} + b), \bar{x} \rangle$ are axially perspective and we have $a \bar{\otimes} b = (z + w)[(\bar{e} + z)(\bar{y} + a) + (\bar{e} + b)h]$. Now the doubly CP triangles $\langle a, y, \bar{y} \rangle$ and $\langle z, e, \bar{e} \rangle$ give us $w + (y + a)(z + e) = w + (\bar{y} + a)(z + \bar{e})$ and the normal triangles $\langle z, b, \bar{e} \rangle$ and $\langle (z + e)(y + a), h(e + b), w \rangle$ produce

$$\begin{aligned} a \otimes b &= (z + w)[(e + z)(y + a) + h(e + b)] \\ &= (z + w)[(\bar{e} + z)(\bar{y} + a) + h(\bar{e} + b)] = a \bar{\otimes} b. \end{aligned}$$

This last part of the proof is, by letting $\bar{y} = x$ (hence $\bar{e} = v$),

$$(3.5) \text{ COROLLARY. } a \otimes b = (z + w)[(y + z)(x + a) + h(v + b)].$$

(3.6) THEOREM. Multiplication is associative.

PROOF. Using $\bar{x} = h(v + c)$ and $\bar{y} = h(v + c \oplus t)$, we obtain

$$\begin{aligned} a \otimes (b \otimes c) &= (a \otimes b) \otimes c \\ &\text{iff } \bar{x} + (\bar{y} + a)(z + (\bar{y} + t)(\bar{x} + b \otimes c)) = \bar{x} + (a \otimes b)_0 \\ &\text{iff } (\bar{y} + a)(z + (\bar{y} + t)(\bar{x} + b \otimes c)) \leq \bar{x} + (a \otimes b)_0. \end{aligned}$$

This is the CP statement of the normal triangles

$$\langle a, z, (a \otimes b)_0 \rangle \quad \text{and} \quad \langle \bar{y}, (\bar{y} + t)(\bar{x} + b \otimes c), \bar{x} \rangle.$$

Therefore \otimes is associative iff $q_2 \leq q_0 + q_1$ where

$$q_2 = (a + z)(\bar{y} + t) = t(z + a), \quad q_1 = h(a + (a \otimes b)_0)$$

and

$$\begin{aligned} q_0 &= (z + (a \otimes b)_0)(\bar{x} + b \otimes c) = (z + (a \otimes b)_0)(y + z)(\bar{x} + b_0) \\ &= (z + (a \otimes b)_0)(x + b + y(v + c)), \end{aligned}$$

by choice of \bar{x} . This inequality follows from the central perspectivity of the triangles $\langle z, a, (a \otimes b)_0 \rangle$ and $\langle b_1, y, x \rangle$.

We are now left with the associativity and commutativity of addition. It is well known that, modulo the multiplicative unit and the distributive laws, associativity implies commutativity (compute $(t \oplus t) \otimes (a \oplus b)$ and use cancellation). In our case the converse also holds modulo the lattice structure.

$$(3.7) \text{ LEMMA. } a \oplus b = (z + t)[(v + w)(y + a) + (v + b)h].$$

PROOF. Consider the triangles $\langle b, w, v \rangle$ and $\langle x, (y + a)(w + b_0), y \rangle$.

(3.8) LEMMA. Let L be an Arguesian lattice; $t.f.a.e$:

(1) For any spanning 3-diamond (x, y, z, t) and $a, b \in D_{(x,y,z,t)}$, $a \oplus b = b \oplus a$.

(2) For any spanning 3-diamond (x, y, z, t) and $a, b, c \in D_{(x,y,z,t)}$, $a \oplus (b \oplus c) = (a \oplus b) \oplus c$.

PROOF. As previously mentioned (2) implies (1), and to show the converse it is sufficient to proof $(b \oplus c) \oplus a = (b \oplus a) \oplus c$ on the diagonal of any spanning 3-diamond. By (3.1), $(b \oplus c) \oplus a = T(t, T(t, b, c), a)$ and we obtain $(b \oplus c) \oplus a = (b \oplus a) \oplus c$ if and only if

$$(y + t)[(y + z)(b_\infty + \bar{c}) + h(z + \bar{a})] = (y + t)[(y + z)(b_\infty + \bar{a}) + h(z + \bar{c})]$$

where $\bar{a} = (y + t)(a_0 + b_\infty)$ and $\bar{c} = (y + t)(c_0 + b_\infty)$. This is a commutativity statement $\bar{c} \oplus \bar{a} = \bar{a} \oplus \bar{c}$ for the spanning 3-diamond $(\bar{x}, \bar{y}, \bar{z}, \bar{t}) = (v, z, b_1, (y + t)(v + b_\infty))$ and the \oplus as in (3.7).

We are now left the proof of commutativity, a proof whose length would ideally be shorter. The first lemma provides the result in Desarguean projective planes by replacing t by an arbitrary b in D and observing that in a projective plane for $a, b \in D$, $a \leq z + b$ or $b \leq z + a$.

$$(3.9) \text{ LEMMA. } a \oplus t = t \oplus a.$$

PROOF. The triangles $\langle z, t, (y + t)(w + v) \rangle$ and $\langle a_0, x, w \rangle$ yield the inequality $(y + a)(w + a_0) \leq (w + a_0)(z + (y + t)(w + v))$. This, in turn, yields the desired result from the triangles

$$\langle y, (y + a)(w + a_0), w \rangle \quad \text{and} \quad \langle v, (y + t)(w + v), t \rangle.$$

Since (y, x, z, t) is a spanning 3-diamond with the same coordinatizing diagonal D , we may define a new addition $\boxplus : D^2 \rightarrow D$ by

$$a \boxplus b = (z + t)(y + (x + a)(w + (x + z)(y + a))).$$

By defining $a_2 = (x + z)(y + a)$ and $u = (x + z)(y + t) = t_2$ we have by symmetry and (3.9) that $a \boxplus t = t \boxplus a$.

(3.10) LEMMA. For $a, b \in D$,

- (1) $h(a_0 + a_2) \leq v + u$,
- (2) $h(b_0 + b_2) = h[(y + a)(w + b_0) + (x + a)(w + b_2)]$,
- (3) $t \oplus a = t \boxplus a$.

PROOF. (1) follows using the triangles $\langle a, x, y \rangle$ and $\langle z, v, u \rangle$; (2), the triply CP triangles $\langle a, x, y \rangle$ and $\langle w, b_2, b_0 \rangle$. To show (3) we have $t \oplus a = t \boxplus a$ iff

$$[x + (y + t)(w + a_0)][y + (x + t)(w + a_2)] \leq z + t.$$

The triangles $\langle x, y, z \rangle$ and $\langle (y + t)(w + a_0), (x + t)(w + a_2), t \rangle$ are normal and, by (1) and (2), centrally perspective.

(3.11) LEMMA. For $a, b \in D$, $a \oplus b = a \boxplus b$.

PROOF. As in the proof of 3.10 (3), we have $a \oplus b = a \boxplus b$ iff

$$[x + (y + a)(w + b_0)][y + (x + a)(w + b_2)] \leq z + t = (a \oplus t) + a.$$

We therefore need to show that the normal triangles

$$\langle x, y, a \oplus t \rangle$$

and

$$\langle (y + a)(w + b_0), (x + a)(w + b_2), a \rangle$$

are axially perspective. This, however, follows from 3.9, 3.10 and the fact that $\langle v, u, w \rangle$ and $\langle y(z + x + b), x(y + z + b), a \rangle$ are centrally perspective.

(3.12) LEMMA. For $a, b \in D$, the following expressions are equal.

- (1) $h[b \oplus t + (x + b)(w + a_2)]$.
- (2) $h[t + a_2]$.
- (3) $h[(x + t)(y + a) + (y + t)(w + v)]$.
- (4) $h[a + (y + t)(w + a_0)]$.
- (5) $h[(x + b)(y + a) + (y + t)(w + b_0)]$.

PROOF. Since all expressions are complements of x in the interval $[0, h]$ we need only show comparability. For (1) and (2), use 3.10 to get the central perspectivity of $\langle y, x, u \rangle$ and $\langle b \oplus t, (x + b)(w + a_2), w \rangle$. For (2) and (3) use $\langle x, w, v \rangle$ and $\langle a_2, t, y \rangle$. For (3) and (4) use $\langle x, w, v \rangle$ and $\langle a, (y + t)(w + a_0), y \rangle$ and finally use $\langle x, w, b_0 \rangle$ and $\langle a, (y + t)(w + a_0), y \rangle$ for (4) and (5).

We are finally at the end.

(3.13) THEOREM. Let L be an Arguesian lattice with spanning 3-diamond (x, y, z, t) . Then $(D; \oplus, z, \otimes, t)$ is $a(n)$ (associative) ring with unit.

PROOF. All that is left is commutativity of \oplus .

$$a \oplus b = b \oplus a \text{ iff } [x + (y + a)(w + b_0)][y + (x + b)(w + a_2)] \leq w + (b \oplus t).$$

Since $\langle x, (x+b)(w+a_2), b \oplus t \rangle$ and $\langle (y+a)(w+b_0), y, w \rangle$ are both normal, this is equivalent to

$$\begin{aligned} (x+b)(y+a) &\leq (x+t \oplus b)(w+b_0) + h((x+b)(w+a_2) + b \oplus t) \\ \text{iff } (x+b)(y+a) &\leq (y+t)(w+b_0) + h((x+b)(w+a_2) + b \oplus t) \\ \text{iff } h[(x+b)(y+a) + (y+t)(w+b_0)] &\leq h[b \oplus t + (x+b)(w+a_2)] \end{aligned}$$

and this follows by (3.12).

We close this section with the analogue of (3.3) for \oplus .

(3.14) **LEMMA.** *For any $\bar{x}, \bar{y} \leq h$ with $\bar{x} + \bar{y} = \bar{x} + w = \bar{y} + w = h$ and $\bar{x} \cdot w = \bar{y} \cdot w = \bar{x} \cdot \bar{y} = 0$,*

$$a \oplus b = (z + w)[\bar{x} + (\bar{y} + a)[w + (\bar{y} + z)(\bar{x} + b)]].$$

PROOF. Using the projectivity $[0, z + t] \xrightarrow{v} [0, h]$ we may assume that $\bar{x} = h(v + c)$ for some $c \in D$. Since (D, \oplus, z) is an Abelian group, we may write $a \oplus b = c \oplus (a \oplus b \ominus c)$ and using (3.7) obtain $a \oplus b = (z + t)[\bar{x} + (w + v)(y + a \oplus b \ominus c)]$. Using the normal triangles

$$\langle \bar{y}, (\bar{y} + z)(\bar{x} + b), \bar{x} \rangle \quad \text{and} \quad \langle a, w, (w + v)(y + a \oplus b \ominus c) \rangle$$

we obtain $a \oplus b = a \oplus b$ if and only if

$$z \leq (w + v)(\bar{x} + b) + h[a + (w + v)(y + a \oplus b \ominus c)].$$

Now for any $d \in D$, $d = c \oplus (d \ominus c)$ and (3.7) give us

$$(w + v)(\bar{x} + d) = (w + v)(y + d \ominus c).$$

Using Δ_r in (2.4) and the triangles $\langle e, (w + v)(y + e \ominus f), w \rangle$ and $\langle x, y, f_0 \rangle$, we obtain $h[e + (w + v)(y + e \ominus f)] = h(f + v)$ for any $e, f \in D$. This produces

$$\begin{aligned} h[a + (w + v)(y + a \ominus (c \ominus b))] &= h[c \ominus b + v] \\ &= h[z + (w + v)(y + b \ominus c)] = h[z + (w + v)(\bar{x} + b)]. \end{aligned}$$

Therefore $a \oplus b = a \oplus b$.

4. von Neumann coordinatization. Although we have worked and will continue to work with the n -diamond concept, it might be of interest to put our results in the framework of von Neumann's classical work. A spanning n -frame in a (modular) lattice L is an independent sequence (a_1, a_2, \dots, a_n) with $\sum_i^n a_i = 1$ and for all i , $a_i \cdot \sum_{j \neq i} a_j = 0$. An n -frame is called homogeneous if there exists $c_{ij} = c_{ji}$ for $i \neq j$ such that $a_i + c_{ij} = a_i + a_j$ and $a_i c_{ij} = a_i a_j (= 0)$ for all $i \neq j$, and $c_{ik} = (a_i + a_k)(c_{ij} + c_{jk})$. If $(a_i, c_{ij}; i \neq j = 1, \dots, n)$ is a homogeneous spanning n -frame then $L_{ij} = \{p: p + a_j = a_i + a_j \text{ and } p a_j = 0\}$. An L -number is a sequence $\alpha \in \prod L_{ij}$ such that, for all i, j, k , $\alpha_{ik} = (a_i + a_k)(c_{ij} + \alpha_{jk}) = (a_i + a_k)(c_{jk} + \alpha_{ij})$. Multiplication of L -numbers is defined by $(\alpha \boxtimes \beta)_{ik} = (a_i + a_k)(\alpha_{ij} + \beta_{jk})$ and addition on each L_{ij} is given by

$$\alpha_{ij} \boxplus \beta_{ij} = (a_i + a_j)((c_{ik} + a_j)(a_k + \alpha_{ij}) + (c_{ik} + \beta_{ij})(a_k + a_j)).$$

Let L be an Arguesian lattice with spanning 3-diamond (x, y, z, t) . By defining $a_1 = z, a_2 = w, a_3 = y, c_{12} = t, c_{23} = x$ and $c_{13} = v = (y + z)(x + t)$, we obtain a

von Neumann homogeneous 3-frame for which $L_{12} = D$. Moreover by (3.7), the von Neumann addition on L_{12} agrees with ours. By (3.5), we have that our multiplication on D is given by the von Neumann formula on L_{12} .

(4.1) LEMMA. *An L -number, α , is uniquely determined by any α_{ij} .*

PROOF. We need only show that the two functions

$$f: [0, z + t] \underset{\wedge}{\overset{x}{\rhd}} [0, y + z] \underset{\wedge}{\overset{t}{\rhd}} [0, x + y] \underset{\wedge}{\overset{v}{\rhd}} [0, z + t],$$

$$g: [0, z + t] \underset{\wedge}{\overset{v}{\rhd}} [0, x + y] \underset{\wedge}{\overset{t}{\rhd}} [0, y + z] \underset{\wedge}{\overset{x}{\rhd}} [0, z + t]$$

agree on D .

Since the triangles $\langle h(t + a_0), v, t \rangle$ and $\langle w, z, u \rangle$ are doubly CP at $y(z + t + a_0)$ (recall $u = (y + t)(x + z)$), we get $f(a) = (z + t)(x + (w + u)(t + a_0))$. Similarly the doubly CP triangles $\langle a, v, x \rangle$ and $\langle w, y, u \rangle$ produce

$$g(a) = (z + w)(x + (y + z)(t + (w + u)(x + a))).$$

Next, the doubly CP triangles $\langle (y + z)(w + u), t, u \rangle$ and $\langle a_0, w, x \rangle$ produce $y + [y + (y + z)(w + u)][w + a_0] = y + (w + u)(x + a)$. This last statement makes the triangles $\langle t, a_0, (w + u)(x + a) \rangle$ and $\langle (y + z)(w + u), w, y \rangle$ doubly CP. The fact that $a \in D$ implies $a(x + y) = 0$, produces

$$x + (w + u)(t + a_0) = x + (y + z)(t + (w + u)(x + a))$$

and hence $f(a) = g(a)$ for all $a \in [0, z + t]$ with $aw = 0$.

We should note that von Neumann showed $f = g$ if L is modular and of order $n \geq 4$. The above is enough for L -numbers in our case but we feel that the full result should hold.

(4.2) LEMMA. *The addition of L -numbers is an L -number.*

PROOF. In our notation we need only show, for $a, b \in D$,

$$(1) a \oplus b = (z + t)[x + (y + z)[(y + t)(w + a_0) + h(t + b_0)]] \text{ and}$$

$$(2) a \oplus b = (z + t)[v + h[b + (w + v)(z + h(v + a))]].$$

For (1), use commutativity of addition and the normal triangles $\langle b, w, x \rangle, \langle y, (y + t)(w + a_0), (y + z)[(y + t)(w + a_0) + h(t + b_0)] \rangle$.

For (2), let $\bar{x} = h[b + (w + v)(z + h(v + a))]$. Since $\bar{x} + w = h$ and $\bar{x} \cdot w = 0$, we can find a $\bar{y} \leq h$ so that the conditions of (3.14) prevail. This gives us (2) if and only if $(\bar{y} + a)(w + (\bar{y} + z)(\bar{x} + b)) \leq v + \bar{x}$. This last statement follows by considering the normal triangles $\langle \bar{y}, (\bar{y} + z)(\bar{x} + b), \bar{x} \rangle$ and $\langle a, w, v \rangle$, and the definition of \bar{x} .

(4.3) LEMMA. *The pointwise multiplication of L -numbers is an L -number agreeing with the von Neumann multiplication.*

PROOF. In our notation, the proof reduces to showing, for $a, b \in D$,

$$(1) a \otimes b = (z + t)[x + (y + z)(a + h(t + b_0))] \text{ and}$$

$$(2) a \otimes b = (z + t)[v + h[b + (y + z)(x + g(a))]].$$

Now (1) follows from considering the normal triangles $\langle y, b_1, x \rangle$ and $\langle a, z, (y+z)(a+h(t+b_0)) \rangle$. For (2) let $\bar{x} = h[b + (y+z)(x+g(a))]$ and note that $\bar{x} + w = h$ and $\bar{x} \cdot w = 0$. By choosing $\bar{y} \leq h$ so that the conditions of (3.4) hold, we have that (2) holds if and only if $(\bar{y} + a)(z + (y+t)(\bar{x} + b)) = (\bar{y} + a)(\bar{x} + v)$ if and only if $(\bar{y} + a)(\bar{x} + v) \leq (\bar{y} + t)(\bar{x} + b) + z$. This last statement follows by considering the normal triangles $\langle \bar{y}, \bar{x}, (\bar{y} + t)(\bar{x} + b) \rangle$, $\langle a, v, z \rangle$ and the definition of \bar{x} .

5. The coordinatization of the hyperplane. In this section we let L be an Arguesian lattice of order $n + 1$, $n \geq 2$, with a given spanning $(n + 1)$ -diamond (x_1, \dots, x_n, z, t) . We will establish a coordinatization function $F: [0, h] \rightarrow \mathcal{L}(D^n)$ which, under reasonable extra conditions, will be a lattice isomorphism onto the sublattice of $\mathcal{L}(D^n)$ generated by all finitely generated submodules of D .

By examining Example B, we see that the following definition should work for all $p \in [0, h]$.

$$\mathbf{a} \in F(p) \quad \text{if and only if} \quad h(z + (1; \mathbf{a})) \leq p.$$

We must show, therefore, several properties of F .

(5.1) LEMMA. For every $\mathbf{a}, \mathbf{b}, \mathbf{c} \in D^n$, $h((1; \mathbf{a}) + (1; \mathbf{b})) = h((1; \mathbf{a} \oplus \mathbf{c}) + (1; \mathbf{b} \oplus \mathbf{c})) = h(z + (1; \mathbf{a} \ominus \mathbf{b}))$.

PROOF. By symmetry it is enough to show that

$$h[(1; \mathbf{a}) + (1; \mathbf{b})] = h[(\bar{x}_1 + a_1 \ominus b_1)A(1) + (\bar{x}_1 + z)B(1)]$$

where $A(1) = \prod_i^{2,n}(\bar{x}_i + a_i) = x_1 + (1; \mathbf{a})$ and $B(1)$ is similarly defined. Now using the several possible expressions for \oplus and \ominus in D we obtain

$$\begin{aligned} a_1 \oplus (b_2 \ominus b_1) &= (a_1 \ominus b_1) \oplus b_2 \\ &= (z + t)[x_1 + (\bar{x}_1 + a_1)(w + (\bar{x}_1 + b_1)(\bar{x}_2 + b_2))] \\ &= (z + t)[x_1 + (\bar{x}_1 + (a_1 \ominus b_1))(w + (\bar{x}_1 + z)(\bar{x}_2 + b_2))]. \end{aligned}$$

The above implies that the triangles $\langle (1; \mathbf{a}), (1; \mathbf{b}), x_1 \rangle$ and $\langle \bar{x}_1, w + \bar{x}_1 \bar{x}_2, (\bar{x}_1 + (a_1 - b_1))(w + (\bar{x}_1 + z)(\bar{x}_2 + b_2)) \rangle$ are (doubly) centrally perspective. Since L is Arguesian we have

$$\begin{aligned} h[(1; \mathbf{a}) + (1; \mathbf{b})] &\leq (\bar{x}_1 + a_1 \ominus b_1)(x_1 + (1; \mathbf{a})) \\ &\quad + (w + (\bar{x}_1 + z)(\bar{x}_2 + b_2))(x_1 + (1; \mathbf{b})) \\ &= (\bar{x}_1 + a_1 \ominus b_1)A(1) + (w + (\bar{x}_1 + z)(\bar{x}_2 + b_2))(\bar{x}_2 + b_2)B(1) \\ &= (\bar{x}_1 + a_1 \ominus b_1)A(1) + (\bar{x}_1 + z)B(1). \end{aligned}$$

(5.2) COROLLARY. For any $p \in [0, h]$, $F(p)$ is a subgroup of D^n .

PROOF. If $\mathbf{a}, \mathbf{b} \in F(p)$ then $h[z + (1; \mathbf{a} \ominus \mathbf{b})] = h[(1; \mathbf{a}) + (1; \mathbf{b})] \leq h[z + (1; \mathbf{a})] + h[z + (1; \mathbf{b})] \leq p$.

The proof that $F(p)$ is closed under scalar multiplication is more intricate. We first observe that $a = k \otimes b$ iff $k \leq a/b$ which implies $w + a/b = z + t$. Moreover for $\mathbf{a}, \mathbf{b} \in D^n$, $\mathbf{a} = k\mathbf{b}$ iff $k \leq \prod_i^{1,n}(a_i/b_i)$ which implies $w + \prod_i^{1,n}(a_i/b_i) = z + t$.

(5.3) THEOREM. $h[z + (1; \mathbf{a})] \leq h[z + (1; \mathbf{b})]$ if and only if $w + \prod(a_i/b_i) = z + t$.

(5.4) COROLLARY. For any $p \in [0, h]$, $F(p)$ is a submodule of ${}_D D^n$.

PROOF OF THEOREM. We have

$$\begin{aligned} h[z + (1; \mathbf{a})] &\leq h[z + (1; \mathbf{b})] && \text{iff } (1, \mathbf{a}) \leq z + (1; \mathbf{b}) \\ &&& \text{iff } (1, \mathbf{a}) \leq A(1)(z + (1; \mathbf{b})) \\ &&& \text{iff } a_1 \leq (z + t)[\bar{x}_1 + A(1)(z + (1; \mathbf{b}))] \end{aligned}$$

and

$$\begin{aligned} w + \prod(a_i/b_i) = z + t & \quad \text{iff } z \leq h(z + t) + \prod(a_i/b_i) \\ & \quad \text{iff } 1 = h + (\bar{x}_1 + (x_1 + a_1)(z + (\bar{x}_1 + t)(x_1 + b_1)))R(1) \\ & \quad \text{iff } 1 = h + (x_1 + a_1)(z + (\bar{x}_1 + t)(x_1 + b_1))(\bar{x}_1 + R(1)) \\ & \quad \text{iff } x_1 + a_1 \leq (\bar{x}_1 + R(1))(z + (\bar{x}_1 + t)(x_1 + b_1)) \\ & \quad \text{iff } a_1 \leq (z + t)[x_1 + (\bar{x}_1 + R(1))(z + (\bar{x}_1 + t)(x_1 + b_1))] \end{aligned}$$

where $R(1) = \prod_i^{2,n}(a_i/b_i)$.

Our theorem is proven once we show

(5.5) THEOREM. For any $(a_2, \dots, a_n), (b_2, \dots, b_n) \in D^{n-1}$ and $c \in D$,

$$\begin{aligned} (z + t)[\bar{x}_1 + A(1)(z + (\bar{x}_1 + c)B(1))] \\ = (z + t)[x_1 + (\bar{x}_1 + R(1))(z + (\bar{x}_1 + t)(x_1 + c))] \end{aligned}$$

where $A(1) = \prod_i^{2,n}(\bar{x}_i + a_i)$, $B(1) = \prod_i^{2,n}(\bar{x}_i + b_i)$ and $R(1) = \prod_i^{2,n}(a_i/b_i)$.

PROOF. We first observe that, using the a_i/b_i ,

$$R(1) = (z + t)[\bar{x}_1 + A(1)(z + (\bar{x}_1 + t)B(1))]^5$$

and that both sides of our desired equation $\lambda = \rho$ are complements of h in the interval $[h(c + zB(1)), h + A(1)(z + B(1))]$. Therefore we need only show $\lambda \leq \rho$.

Since the triangles

$$\langle w, A(1)[z + (\bar{x}_1 + c)B(1)], x_1 \rangle$$

and

$$\langle z, \bar{x}_1, (\bar{x}_1 + R(1))(z + (\bar{x}_1 + t)(x_1 + c)) \rangle$$

are normal, we have $\lambda \leq \rho$ if and only if $p_2 \leq p_0 + p_1$ where

$$\begin{aligned} p_2 &= (x_1 + z)[w + A(1)[z + (\bar{x}_1 + c)B(1)]], \\ p_1 &= (w + x_1)(z + \bar{x}_1 + R(1))(z + (\bar{x}_1 + t)(x_1 + c)) \\ &= h(z + t + x_1)(z + (\bar{x}_1 + t)(x_1 + c))(z + \bar{x}_1 + R(1)), \\ p_0 &= [x_1 + A(1)[z + (\bar{x}_1 + c)B(1)]][\bar{x}_1 + R(1)][z + \bar{x}_1 + (\bar{x}_1 + t)(x_1 + c)] \\ &= A(1)[x_1 + z + (\bar{x}_1 + c)B(1)][\bar{x}_1 + A(1)[z + (\bar{x}_1 + t)B(1)]] \\ &= A(1)[z + (\bar{x}_1 + t)B(1)]. \end{aligned}$$

⁵See Appendix 3.

Therefore $\lambda \leq \rho$ iff $p_2 \leq p_0 + p_1$ iff

$$\begin{aligned} & (\bar{x}_1 + z)[w + A(1)[z + (\bar{x}_1 + c)B(1)]] \\ & \leq h[z + (\bar{x}_1 + t)(x_1 + c)] + A(1)[z + (\bar{x}_1 + t)B(1)]. \end{aligned}$$

Now consider the normal triangles $\langle (\bar{x}_1 + z)(\bar{x}_2 + t), w, h(z + (\bar{x}_1 + t)(x_1 + c)) \rangle$ and $\langle z, A(1)[z + (\bar{x}_1 + c)B(1)], A(1)[z + (\bar{x}_1 + t)B(1)] \rangle$. Since L is Arguesian we obtain $\lambda \leq \rho$ iff $q_2 \leq q_0 + q_1$ where

$$\begin{aligned} q_2 &= [z + A(1)][z + (\bar{x}_1 + c)B(1)][w + (\bar{x}_1 + z)(\bar{x}_2 + t)], \\ q_1 &= [z + A(1)][z + (\bar{x}_1 + t)B(1)][(\bar{x}_1 + z)(\bar{x}_2 + t) + h(z + (\bar{x}_1 + t)(x_1 + c))], \\ q_0 &= h(z + t + (\bar{x}_1 + t)(x_1 + c))\{A(1)[z + (\bar{x}_1 + c)B(1)] \\ & \quad + A(1)[z + (\bar{x}_1 + t)B(1)]\} \\ &= x_1[z + (\bar{x}_1 + c)B(1) + A(1)[z + (\bar{x}_1 + t)B(1)](\bar{x}_1 + t + c). \end{aligned}$$

Since $q_0 \leq x_1 \leq z + A(1)$ and $q_1, q_2 \leq (\bar{x}_1 + c)B(1) + [z + A(1)][z + (\bar{x}_1 + t)B(1)]$ we get $q_2 \leq q_0 + q_1$ iff

$$\begin{aligned} q_2 &\leq x_1(\bar{x}_1 + t + c) \\ &+ [z + (\bar{x}_1 + t)B(1)][(\bar{x}_1 + z)(\bar{x}_2 + t) + h(z + (\bar{x}_1 + t)(x_1 + c))]. \end{aligned}$$

Finally consider the triangles

$$\langle z, (\bar{x}_1 + c)B(1), (\bar{x}_1 + t)B(1) \rangle$$

and

$$\langle (\bar{x}_1 + z)(\bar{x}_2 + t), w, h(z + (\bar{x}_1 + t)(x_1 + c)) \rangle.$$

By defining, for $i = 2, \dots, n$,

$$c_i = b_i \ominus c = (z + t)\{\bar{x}_i + (\bar{x}_1 + z)[w + (\bar{x}_1 + c)(\bar{x}_i + b_i)]\}$$

and noting that $(\bar{x}_1 + z)[w + (\bar{x}_1 + c)B(1)] \leq C(1)$ and $h[z + (\bar{x}_1 + t)(x_1 + c)] = h[(\bar{x}_1 + t)B(1) + (\bar{x}_1 + z)C(1)]$, by (5.1) we have that they are centrally perspective. Since L is Arguesian, we have $r_2 \leq r_0 + r_1$ where $q_2 = [z + A(1)]r_2$,

$$\begin{aligned} r_1 &= [z + (\bar{x}_1 + t)B(1)][(\bar{x}_1 + z)(\bar{x}_2 + t) + h(z + (\bar{x}_1 + t)(x_1 + c))], \\ r_0 &= h[z + t + (\bar{x}_1 + t)(x_1 + c)]\{B(1)(\bar{x}_1 + c) + B(1)(\bar{x}_1 + t)\} \\ &\leq x_1(\bar{x}_1 + t + c). \end{aligned}$$

Therefore $\lambda = \rho$ and the theorem is proven.

(5.6) COROLLARY. $F: [0, h] \rightarrow \mathcal{L}(D^n)$ is $a(n)$ (arbitrary) meet-preserving function with $\langle \mathbf{a} \rangle \leq F(h[z + (1; \mathbf{a})])$ for every $\mathbf{a} \in D^n$.

In order to strengthen this corollary and make F at least a lattice homomorphism we seem to need the extra property of *upper complementability* for joins of elements in our spanning $(n + 1)$ -diamond: if $g = \Sigma M$ for some $M \subseteq \{z, t, x_1, \dots, x_n\}$ and $p + g = 1$ then there exists $s \in L$, $s \leq p$, $s + g = 1$ and $s \cdot g = 0$. This property is called FC(b) in [18] and holds for all geometric elements of a primary lattice [15, 5.2]. It also holds in Example B.

(5.7) LEMMA. *Let L satisfy (UC); then $a \leq z + b$ if and only if $w + a/b = z + t$ if and only if $a = k \otimes b$ for some $k \in D$.*

PROOF. Since $w + a/b = w + a(z + b)$, the first equivalence is clear. If $w + a/b = z + t$ then $h + a/b = 1$ and there exists $k \in A$ with $k \leq a/b \leq z + t$. Therefore $k \in D$ and $a = k \otimes b$.

(5.8) LEMMA. *Let L satisfy (UC); then for $\mathbf{a}, \mathbf{b} \in D^n$, $h[z + (1; \mathbf{a})] \leq h[z + (1; \mathbf{b})]$ if and only if there exists a $k \in D$ with $\mathbf{a} = k\mathbf{b}$.*

PROOF. (5.7) and (5.3).

(5.9) THEOREM. *If L satisfies (UC), then $F: [0, h] \rightarrow \mathcal{L}({}_D D^n)$ is a lattice homomorphism with $F(h(z + (1; \mathbf{a}))) = \langle \mathbf{a} \rangle$ for every $\mathbf{a} \in D^n$.*

PROOF. We need only show that F preserves joins. If $\mathbf{a} \in F(p + q)$ then $h(z + (1; \mathbf{a})) \leq p + q$. Let $s = (z + p)((1; \mathbf{a}) + q)$.

$$\begin{aligned} h + s &= h + p + q + s = h + (z + p + q)((1; \mathbf{a}) + p + q) \\ &= h + (1; \mathbf{a})(z + p + q) = h + (1; \mathbf{a}) = 1. \end{aligned}$$

By (UC), there exists $\mathbf{b} \in D^n$ with $(1; \mathbf{b}) \leq s$. Now

$$\begin{aligned} h(z + (1; \mathbf{b})) &\leq h(z + s) \leq p, \\ h(z + (1; \mathbf{a} \oplus \mathbf{b})) &= h((1; \mathbf{a}) + (1; \mathbf{b})) \leq h((1; \mathbf{a}) + s) \leq q. \end{aligned}$$

Therefore $\mathbf{a} = \mathbf{b} \oplus (\mathbf{a} \ominus \mathbf{b}) \in F(p) + F(q)$.

(5.10) COROLLARY. *If L satisfies (UC) and either $[0, h]$ is simple, or any proper congruence on $[0, h]$ must identify two distinct elements of the form $h[z + (1; \mathbf{a})]$, $\mathbf{a} \in D^n$, then $F: [0, h] \rightarrow \mathcal{L}({}_D D^n)$ is a lattice isomorphism onto a sublattice of $\mathcal{L}({}_D D^n)$ containing all finitely generated submodules of ${}_D D^n$.*

If L is primary, then so is $[0, h]$ and any primary lattice is simple. Therefore F will be an embedding into $\mathcal{L}({}_D D^n)$ and in fact an isomorphism. If L is complemented then F is injective iff $F^{-1}(0) = \{0\}$. This will occur if and only if $F|_{[0, w]}$ is injective since $F(p) = F(0)$ implies $F(w(\bar{x}_i + p)) = F(0)$ for every $i = 1, \dots, n$ and $p > 0$ forces at least one of $w(x_i + p) > 0$. Since $F|_{[0, w]}$ is injective if L is complemented we again get F to be an embedding. Therefore our theorem covers the known results as far as the hyperplane is concerned.

6. Concluding remarks. Following von Neumann's Case II and using extensions of (5.1) and (5.5) one can easily complete the isomorphism $L \simeq \bar{\mathcal{L}}({}_D D^n)$ when L is a complemented Arguesian lattice of order $n \geq 3$. By repeating three sections of Jónsson and Monk, one can produce the desired isomorphism in case L is a primary Arguesian lattice of order $n \geq 3$. We omit these important proofs, however, since we are unable to present any unified perspective on them. Each proof, at present, requires a different structural analysis. Whether a common generalization does indeed exist is, in our view, a challenging open problem.

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Appendices. We include here, at the suggestion of the referee, some modular calculations that were omitted from the main text. Hopefully these will aid the reader's understanding of the material.

$$1. T_{(p,q)}(a, m, b) \in D,$$

$$\begin{aligned} w + T_{(p,q)}(a, m, b) = z + t & \text{ iff } a \leq w + T_{(p,q)}(a, m, b) \\ & \text{ iff } a \leq (p + w)(q + a) + (p + q)[z + (q + t)(p + m)] + (q + z)(p + b) \\ & \text{ iff } a \leq q + (p + q)[z + (q + t)(p + m)] + (q + z)(p + b). \end{aligned}$$

Now

$$\begin{aligned} q + (q + z)(p + b) &= (q + z)(q + p + w + b) = q + z, \\ q + (q + t)(p + m) &= (q + t)(q + p + w + m) = q + t. \end{aligned}$$

These give

$$\begin{aligned} q + (p + q)[z + (q + t)(p + m)] + (q + z)(p + b) \\ &= q + z + (p + q)[z + (q + t)(p + m)] \\ &= (p + q + z)(q + z + t) \geq a \end{aligned}$$

as desired.

$$\begin{aligned} w \cdot T_{(p,q)}(a, m, b) &= (p + q) \cdot T_{(p,q)}(a, m, b) \\ &= (z + t)[p + q[(p + q)(z + (q + t)(p + m)) + (q + z)(p + b)]] \\ &= (z + t)[p + q[z + (q + t)(p + m) + (p + q)(q + z)(p + b)]] \\ &= (z + t)[p + q[z + (q + t)(p + m)]] \\ &= (z + t)(p + q)(p + m) = 0. \end{aligned}$$

2. LEMMA 2.5(3). For $a, b, c \in D$ we have

$$\begin{aligned} z + b \otimes a &= (z + t)(z + x + (y + b)(z + a_1)) \\ &= (z + t)(z + x + a_1(y + z + b)) \\ &= z + (z + t)(x + (x + a)(y + t)(y + z + b)) \\ &= z + (z + t)(x + a)(x + y + t(z + b)) \\ &= z + a(w + t(z + b)) \leq z + a. \end{aligned}$$

Similarly $z \cdot a \otimes c = z(a + w(t + zb)) \geq za$.

Now if $a \in \text{Inv}(D)$ then by (2), t/a and $a \setminus t \in D$. Easy computations give, if $a \in \text{Inv}(D)$, $(t/a) \otimes a = t = a \otimes (a \setminus t)$. Conversely if $b \otimes a = t = a \otimes c$ for some $b, c \in D$ then $z + a \leq z + t = z + b \otimes a \leq z + a$ and $0 \leq z \cdot a \leq z \cdot a \otimes c = z \cdot t = 0$. Therefore $a \in \text{Inv}(D)$.

3. THEOREM 5.5. For each $i = 2, \dots, n$ we have, using (\bar{x}_i, \bar{x}_1) ,

$$\begin{aligned} a_i/b_i &= (z + t)(\bar{x}_1 + (\bar{x}_i + a_i)(z + (\bar{x}_1 + t)(\bar{x}_i + b_i))) \\ &= (z + t)(x_i + (\bar{x}_i + a_i)(z + (\bar{x}_1 + t)(\bar{x}_i + b_i))). \end{aligned}$$

Moreover, for $i \neq j$, $x_i \leq \bar{x}_j \cdot \bar{x}_1 \leq (\bar{x}_j + a_j)(z + (\bar{x}_1 + t)(\bar{x}_j + b_j))$. Therefore

$$\begin{aligned} R(1) &= (z + t) \cdot \prod_i^{2,n} (x_i + (\bar{x}_i + a_i)(z + (\bar{x}_1 + t)(\bar{x}_i + b_i))) \\ &= (z + t) \left[\sum_i^{2,n} x_i + \prod_i^{2,n} (\bar{x}_i + a_i)(z + (\bar{x}_1 + t)(\bar{x}_i + b_i)) \right] \\ &= (z + t) \left[\bar{x}_1 + A(1) \prod_i^{2,n} (z + (\bar{x}_1 + t)(\bar{x}_i + b_i)) \right] \\ &= (z + t) [\bar{x}_1 + A(1)(z + (\bar{x}_1 + t)B(1))]. \end{aligned}$$

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